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Exact bright soliton solution for a family of coupled higher-order nonlinear Schrödinger equation in inhomogeneous optical fiber media

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Abstract. In this paper, we present a family of coupled higher-order nonlinear Schrödinger equation describing the optical soliton pulse propagating in inhomogeneous optical fiber media. The exact N-soliton solution and its characteristics of stabilities and novel elastic collision properties are studied in detail. As an example, we give the relative numerical evolutions by a soliton control system to discuss the pulses propagation characteristics.

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1 Introduction

Hasegawa and Tappert [1,2] theoretically predicted the possibility of propagation of envelope solitons in optical fibers and it was experimentally demonstrated by Mollenauer et al. [3] in 1980. Since then, numerous interesting research results, both theoretical and experimental have been reported in the field of optical solitons, as they are very useful in high speed digital optical fiber communication [4]. One of the most important models in this area is the nonlinear Schrödinger equation (NLS) equation, which describes the propagation of picosecond light pulses in optical fiber. And when to describe the propagation of femtosecond light pulses due to their extensive applications to telecommunication and ultrafast signal-routing systems, the governing equation is higher-order nonlinear Schrödinger (HNLS) equation and in recent years, many authors have analyzed the HNLS equation from different points of view and some interesting results have also been obtained [5–9]. In recent years, the study of coupled NLS (CNLS) equations is receiving a great deal of attention due to their appearance as modeling equations in diverse areas of physics like nonlinear optics, optical birefringent effects, optical coupler, coupled mode approach, Bose-Einstein condensates, etc. [10]. To be specific, soliton type pulse propagation in multimode fibers and in fiber arrays is also governed by a set of CNLS equations. On the other hand, the recent studies on the coherent and incoherent beam propagation in photorefractive media, which can exhibit high nonlinearity with extremely low optical power, necessitate intense study of CNLS equations [11].

In the area of optical communication, wavelength-division multiplexing (WDM) using solitons is necessary to propagate more channels simultaneously and also to increase the transmission capacity of the communication system where at least two optical fields are to be transmitted and the system is governed by 2-CNLS equations [11–13]. In 1974, Manakov [14] proposed a coupled version of the NLSE (CNLSE) by considering the left- and right-polarized modes of the propagating electromagnetic wave and presented the linear eigenvalue problem associated with the CNLS equations and the soliton solutions using the inverse scattering transform (IST). Later in 1999, Porsezian et al. [15] have generalized the 2×2 AKNS method to the $(2N+1)\times(2N+1)$ eigenvalue problem of N-coupled HNLS equations and in [4] the author investigated the exact dark soliton solutions for a family of N-coupled HNLS equations. Then in 2004, we obtained a new combined solitary wave solution of the 2-coupled HNLS equations [16].

The concept of soliton control and soliton management is a new and important development in the application of solitons. Picosecond soliton control which is described by the NLS equation with variable coefficients has been extensively studied theoretically because of its potential value [17–20]. The studies of femtosecond soliton control, which is governed by the HNLS equation with variable coefficients, also have been widespread [21,22]. And for handling femtosecond optical soliton control in more channels in inhomogeneous optical fiber media, in this paper, we

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will consider the 2-coupled HNLS equations with variable coefficients.

The paper is organized in the following sequence. In Section 2 the 2-coupled HNLS equation with variable coefficients is introduced and the linear eigenvalue problem is given and through which the exact N-soliton solution was obtained by the well-known Darboux transformation. Then analytical investigation of the pulse interaction and numerical evolutions for the pulse propagating in a soliton control system were given. Section 3 is the concluding one.

2 Models and analysis

The 2-coupled HNLS equation with variable coefficients may be written as

$$E_{jz}+\frac{1}{2}iD_2(z)E_{jtt}+iF(z)\left(\sum_{n=1}^{2}E_nE_n^*\right)E_j+D_3(z)E_{jttt}+M(z)\left(\sum_{n=1}^{2}E_nE_n^*\right)E_{jt}+N(z)\left(\sum_{n=1}^{2}E_{nt}E_n^*\right)E_j+G(z)E_j=0 \quad (1)$$

where $E_j\,(j=1,2)$ is the 2-component electric field. z and t denote the direction of propagation and time variable respectively. The limitary real functions of $D_2(z)$, $D_3(z)$, $F(z)$ and $G(z)$ are respectively the variable second-order dispersion, third-order dispersion, cross-phase modulation, amplification or absorption coefficients and the functions of $M(z)$ and $N(z)$ describe the effects of Kerr nonlinearity and simulated Raman scattering. Here * denotes complex conjugate, and the subscripts in z and t denote derivatives with respect to z and t.

Now let us consider the linear eigenvalue problem given as

$$\Phi_t=U\Phi,\quad \Phi_z=V\Phi,\quad \Phi=(\phi_1,\phi_2,\phi_3)^T \quad (2)$$

where the superscript T denotes matrix transpose and

$$U=\begin{pmatrix}-i\lambda & mE_1 & mE_2\\ -mE_1^* & i\lambda & 0\\ -mE_2^* & 0 & i\lambda\end{pmatrix}$$

$$V=-iD_2\left(4\lambda^3+\lambda^2\right)\begin{pmatrix}-1&0&0\\0&1&0\\0&0&1\end{pmatrix}-iD_2\left(4\lambda^2+\lambda\right)\times\begin{pmatrix}0&-imE_1&-imE_2\\ imE_1^*&0&0\\ imE_2^*&0&0\end{pmatrix}+\frac{1}{2}iD_2(4\lambda+1)\times\begin{pmatrix}-m^2E_1E_1^*-m^2E_2E_2^* & -mE_{1\,t} & -mE_{2\,t}\\ -mE_{1t}^* & m^2E_1E_1^* & m^2E_2E_1^*\\ -mE_{2t}^* & m^2E_1E_2^* & m^2E_2E_2^*\end{pmatrix}-iD_2\begin{pmatrix}A_{11}&A_{12}&A_{13}\\A_{21}&A_{22}&A_{23}\\A_{31}&A_{32}&A_{33}\end{pmatrix} \quad (3)$$

with

$$\begin{aligned}
A_{11}&=-im^2\left(E_1E_{1t}^*-E_{1\,t}E_1^*+E_2E_{2t}^*-E_{2\,t}E_2^*\right)\\
A_{12}&=imE_{1\,tt}+2im^3\left(E_1E_1^*+E_2E_2^*\right)E_1,\\
A_{13}&=imE_{2\,tt}+2im^3\left(E_1E_1^*+E_2E_2^*\right)E_2\\
A_{21}&=-imE_{1tt}^*-2im^3\left(E_1E_1^*+E_2E_2^*\right)E_1^*,\\
A_{31}&=-imE_{2tt}^*-2im^3\left(E_1E_1^*+E_2E_2^*\right)E_2^*\\
A_{22}&=im^2\left(E_1E_{1t}^*-E_{1,\,t}E_1^*\right),\\
A_{32}&=im^2\left(E_1E_{2t}^*-E_{1,\,t}E_2^*\right)\\
A_{23}&=im^2\left(E_2E_{1t}^*-E_{2,\,t}E_1^*\right),\\
A_{33}&=im^2\left(E_2E_{2t}^*-E_{2,\,t}E_2^*\right)
\end{aligned}$$

where $m=\sqrt{F}/D_2$. The compatibility condition $U_z-V_t+[U,V]=0$ gives equation (1) under the restriction conditions below:

$$D_3=-D_2,\quad M=N=-3F,\quad G(z)=\frac{F_zD_2-FD_{2,z}}{2FD_2}. \quad (4)$$

The Lax pair (3) and the restriction conditions (4) give the integrable conditions of equation (1). The conditions can also be obtained by the following transformation:

$$x=\int_0^z(\varsigma)d\varsigma,\quad \frac{dx}{dz}=D_2(z),$$

$$E_j(z,t)=q_j(x,t)\sqrt{\frac{D_2(z)}{F(z)}}. \quad (5)$$

Introducing equation (5) into equation (1) and using the condition (4), we can obtain

$$q_{jx}+\frac{1}{2}iq_{jtt}+i\left(\sum q_nq_n^*\right)q_j-q_{jttt}-3\left(\sum q_nq_n^*\right)q_{jt}-3\left(\sum q_{nt}q_n^*\right)q_j=0. \quad (6)$$

This is the well-known integrable form of coupled Hirota equations [23].

By far, we have find the integrable condition or one may call it solvable condition of equation (1). To find its exact soliton solutions, we have used a simple, straightforward Darboux transformation based on the linear eigenvalue problem (2) [24] and the detailed processes were given in Appendix.

According to the standard procedure of Darboux transformation, we can obtain the N-soliton solution of equation (1)

$$E_1'[N]=E_1+2i\sqrt{\frac{D_2}{F}}\times\sum_{n=1}^{N}\frac{-\left(\lambda_n-\lambda_n^*\right)\phi_2^*[n,\lambda_n]\phi_1[n,\lambda_n]}{\left|\phi_1[n,\lambda_n]\right|^2+\left|\phi_2[n,\lambda_n]\right|^2+\left|\phi_3[n,\lambda_n]\right|^2}$$

$$E_2'[N]=E_2+2i\sqrt{\frac{D_2}{F}}\times\sum_{n=1}^{N}\frac{-\left(\lambda_n-\lambda_n^*\right)\phi_3^*[n,\lambda_n]\phi_1[n,\lambda_n]}{\left|\phi_1[n,\lambda_n]\right|^2+\left|\phi_2[n,\lambda_n]\right|^2+\left|\phi_3[n,\lambda_n]\right|^2} \quad (7)$$

where

$$\phi_k[j+1,\,\lambda_{j+1}] = \left(\lambda_{j+1}-\lambda_j^*\right)\,\phi_k[j,\,\lambda_{j+1}] - \frac{P_j}{Q_j}\left(\lambda_j-\lambda_j^*\right)\,\phi_k[j,\,\lambda_j]$$

$$P_j = \phi_1^*[j,\,\lambda_j]\phi_1[j,\,\lambda_{j+1}] + \phi_2^*[j,\,\lambda_j]\phi_2[j,\,\lambda_{j+1}] + \phi_3^*[j,\,\lambda_j]\phi_3[j,\,\lambda_{j+1}]$$

$$Q_j = \left|\phi_1[j,\,\lambda_j]\right|^2 + \left|\phi_2[j,\,\lambda_j]\right|^2 + \left|\phi_3[j,\,\lambda_j]\right|^2$$

$k = 1, 2, 3;\; j = 1, 2, ..., n-1;\; n = 2, 3, ..., N$ and $(\phi_1[j,\,\lambda_j], \phi_2[j,\,\lambda_j], \phi_3[j,\,\lambda_j])^T$ is the eigenfunction of the 2-coupled form of equation (1) corresponding to λ_j. Substituting the zero solution of equation (1) into equation (7), we can systematically obtain multisoliton solutions for equation (1). Here we present only one- and two-soliton solutions in explicit forms.

By setting $N = 1$, taking zero seed solution in equation (7) and setting the complex spectral parameter $\lambda_1 = (\eta_1 + i\xi_1)/2$, we find that the one-soliton solution is of the form

$$E_j' = -\sqrt{\frac{D_2}{F}}\,i\left(\lambda_1-\lambda_1^*\right)\,\frac{\varepsilon_j\exp\left(2i\varphi_1\right)}{\cosh\left(2\vartheta_1+T_0\right)},\quad (j=1,2) \tag{8}$$

where $|\varepsilon_1|^2 + |\varepsilon_2|^2 = 1$, T_0 is an arbitrary real constant and

$$\vartheta_1 = \frac{1}{2}\xi_1(3\eta_1^2-\xi_1^2+\eta_1)\int D_2(z)dz + \frac{1}{2}\xi_1 t$$
$$\varphi_1 = \left(\frac{1}{4}\xi_1^2-\frac{1}{4}\eta_1^2+\frac{3}{2}\xi_1^2\eta_1-\frac{1}{2}\eta_1^3\right)\int D_2(z)dz - \frac{1}{2}\eta_1 t. \tag{9}$$

From above expressions, we can see that the imaginary part ξ_1 of the spectral parameter λ_1 is mainly dependent on pulse width and its real part η_1 describes the frequency shift. The phase shift is related to both real and imaginary parts of the spectral parameter λ_1, and the initial position is determined by the parameter T_0. From the expressions (8) and (9), we can find that the velocity of the soliton is determined by $(3\eta_1^2-\xi_1^2+\eta_1)D_2(z)$, which depends on the variable $D_2(z)$ except for the spectral parameter λ_1. Thus, we can control the velocity of the soliton by managing the variable second-order dispersion parameter $D_2(z)$ in optical communication systems [19].

When $N = 2$, taking the zero seed solution and setting the spectral parameter $\lambda_k = (\eta_k + i\xi_k)/2$, $(k = 1, 2)$, from equation (7), we can obtain the two-soliton solution

$$E_j'\left[2\right] = \chi_j\sqrt{\frac{D_2}{F}}\frac{H}{K},\quad (j=1,2)\,, \tag{10}$$

where

$$|\chi_1|^2 + |\chi_2|^2 = 1$$

$$H = a_1\exp\left(2i\varphi_1\right)\cosh\left(2\vartheta_2\right) + a_3\exp\left(2i\varphi_2\right)\cosh\left(2\vartheta_1\right) + ia_2\left(\exp\left(2i\varphi_1\right)\sinh\left(2\vartheta_2\right) - \exp\left(2i\varphi_2\right)\sinh\left(2\vartheta_1\right)\right) \tag{11}$$

$$K = b_1\cosh\left(2\vartheta_1+2\vartheta_2\right) + b_2\cosh\left(2\vartheta_1-2\vartheta_2\right) + b_3\cos\left(2\varphi_1-2\varphi_2\right) \tag{12}$$

with

$$a_1 = \xi_1\left(\left(\xi_1^2-\xi_2^2\right)+\left(\eta_2-\eta_1\right)^2\right),$$
$$a_2 = 2\xi_1\xi_2\left(\eta_2-\eta_1\right),$$
$$a_3 = \xi_2\left(\left(\xi_2^2-\xi_1^2\right)+\left(\eta_2-\eta_1\right)^2\right),$$
$$b_1 = \left(\frac{1}{2}\left(\xi_2-\xi_1\right)^2+\frac{1}{2}\left(\eta_2-\eta_1\right)^2\right),$$
$$b_2 = \left(\frac{1}{2}\left(\eta_2-\eta_1\right)^2+\frac{1}{2}\left(\xi_1+\xi_2\right)^2\right),$$
$$b_3 = -2\xi_1\xi_2$$
$$\vartheta_k = \frac{1}{2}\xi_k\left(3\eta_k^2-\xi_k^2+\eta_k\right)\int D_2\left(z\right)\,dz + \frac{1}{2}\xi_k t \tag{13}$$
$$\varphi_k = \left(\frac{1}{4}\xi_k^2-\frac{1}{4}\eta_k^2+\frac{3}{2}\xi_k^2\eta_k-\frac{1}{2}\eta_k^3\right)\times\int D_2\left(z\right)\,dz - \frac{1}{2}\eta_k t,\quad k=1,2. \tag{14}$$

Based on this exact two-soliton solution, we can conveniently analyze the transmission properties of two femtosecond optical solitons in inhomogeneous systems. From the expressions of ϑ_k and φ_k, we can clearly see that similarly to the above results for the one-soliton solution, the velocity of each soliton in the two-soliton solution (10) is determined by $\left(3\eta_k^2-\xi_k^2+\eta_k\right)D_2\left(z\right)$, which is determined by both the variable parameter $D_2\left(z\right)$ and the spectral parameter λ_k. The pulse width and frequency shift are respectively determined by the imaginary part ξ_k and real part η_k of the spectral parameter λ_k. The pulse phase is also determined by both the variable parameter $D_2\left(z\right)$ and the spectral parameter λ_k.

Because the solutions include distributed functions, thus by choosing the form most approximate to the real state, one can explain different types of soliton control or dispersion management. Here as an example, we consider a soliton control system with the second-order dispersion parameter

$$D_2\left(z\right) = d\left[1+c_1\sin\left(\sigma z\right)\right], \tag{15}$$

and the crossphase modulation parameter

$$F\left(z\right) = f\left[1+c_2\sin\left(\sigma z\right)\right]\exp\left(-gz\right), \tag{16}$$

where d, σ, c_k and g are the parameters of the control system. Then the amplification or absorption coefficients can be obtained as

$$G\left(z\right) = \frac{g}{2} + \frac{\sigma\left(c_1-c_2\right)\cos\left(\sigma z\right)}{2\left[1+c_1\sin\left(\sigma z\right)\right]\left[1+c_2\sin\left(\sigma z\right)\right]}. \tag{17}$$

We can find that when $c_1 = c_2$, we have $G\left(z\right) = g/2$. Thus the maximal amplitude of the soliton will undergo

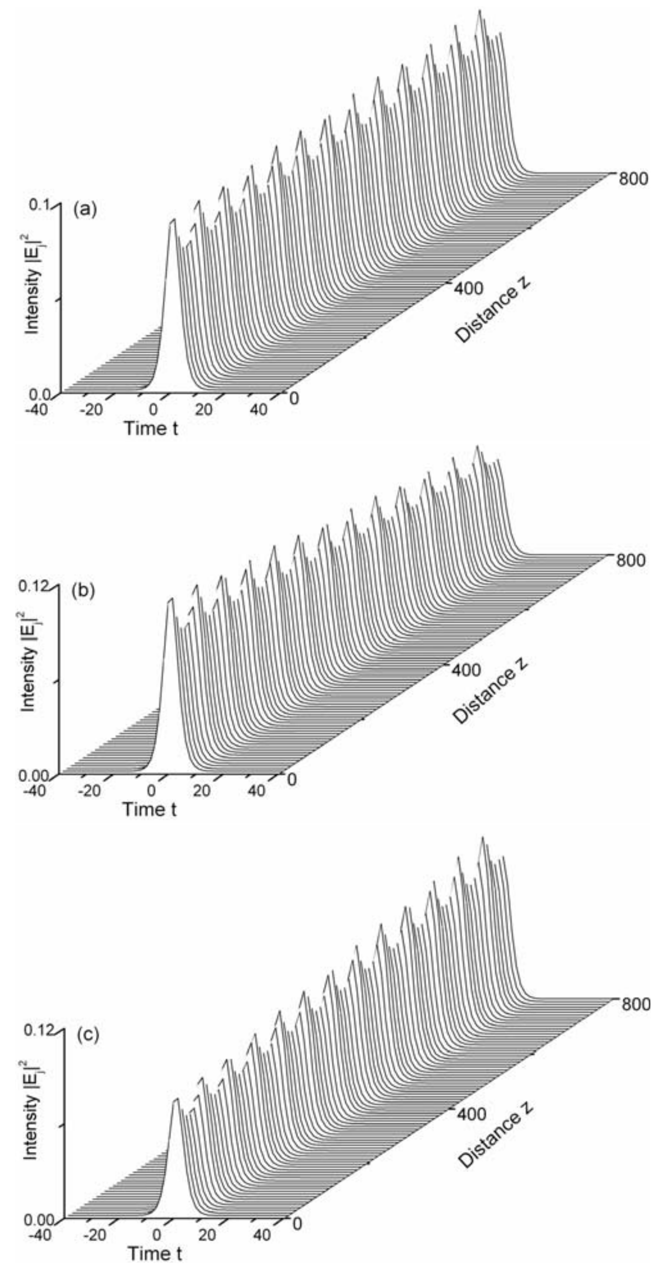

Fig. 1. Pulse evolution plots of one-soliton solution when $d = -1$, $f = -2$, $\sigma = 2$, $T_0 = 0$, $\xi_1 = 0.56$, $\eta_1 = 0.2$, $\varepsilon_1 = \varepsilon_2 = \sqrt{2}/2$, $c_1 = 0.4$, $c_2 = 0.5$, and (a) $g = 0$, (b) $g = -0.01$, (c) $g = 0.01$.

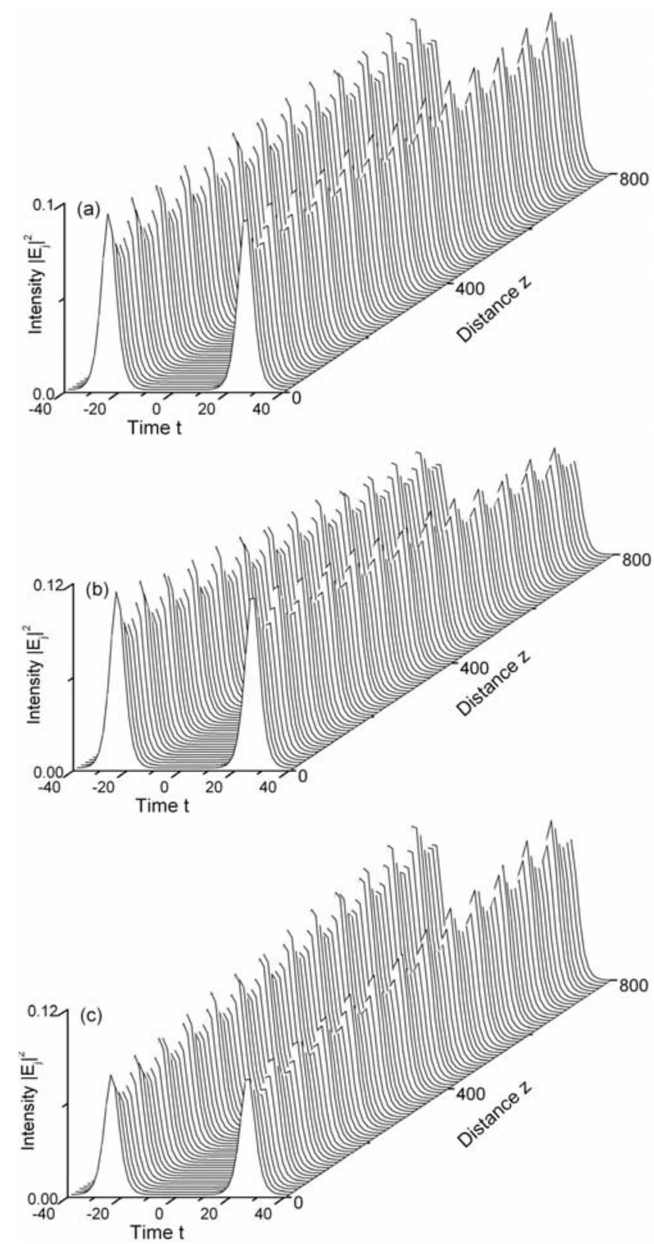

Fig. 2. Pulse evolution plots of two-soliton solution when $d = -1$, $f = -2$, $\sigma = 2$, $T_0 = 0$, $\xi_1 = 0.560$, $\eta_1 = 0.2$, $\xi_2 = -0.562$, $\eta_2 = 0.2$, $\chi_1 = \chi_2 = \sqrt{2}/2$, $c_1 = 0.4$, $c_2 = 0.5$, and (a) $g = 0$, (b) $g = -0.01$, (c) $g = 0.01$.

the increase ($g > 0$) or decrease ($g < 0$) when propagating along the optical fiber. But for the general case, $c_1 \neq c_2$, the maximal amplitude of the pulse will increase ($g > 0$) or decrease ($g < 0$) exponentially and periodically. The results were illustrated in Figure 1 for one-soliton solution and in Figure 2 for two-soliton solution. We can see from the figures that the pulses propagate very stable with all the characteristics mentioned above. The contour plots of the one- and two-soliton solution as illustrated in Figure 3 for the case of $g = 0$. Since we have known that the velocity of each soliton in the two-soliton solution (10) is determined by $\left(3\eta_k^2 - \xi_k^2 + \eta_k\right) D_2\left(z\right)$, this will leads to a change of the center position of the soliton along the propagation direction of the fiber and thus it provides a way for us to design a fiber system to control the soliton velocity.

It is easy to understand that the above stable cases are only common examples under given parameters conditions. When the parameters conditions changed, the pulse interaction is inevitably. So it is necessary for us to analyze asymptomatically the interacting solitons. To do this, the lengthy forms of the above expressions of H and K with ϑ_k and φ_k are useful. And a more explicit manner may be obtained by observing the expression of the two-soliton

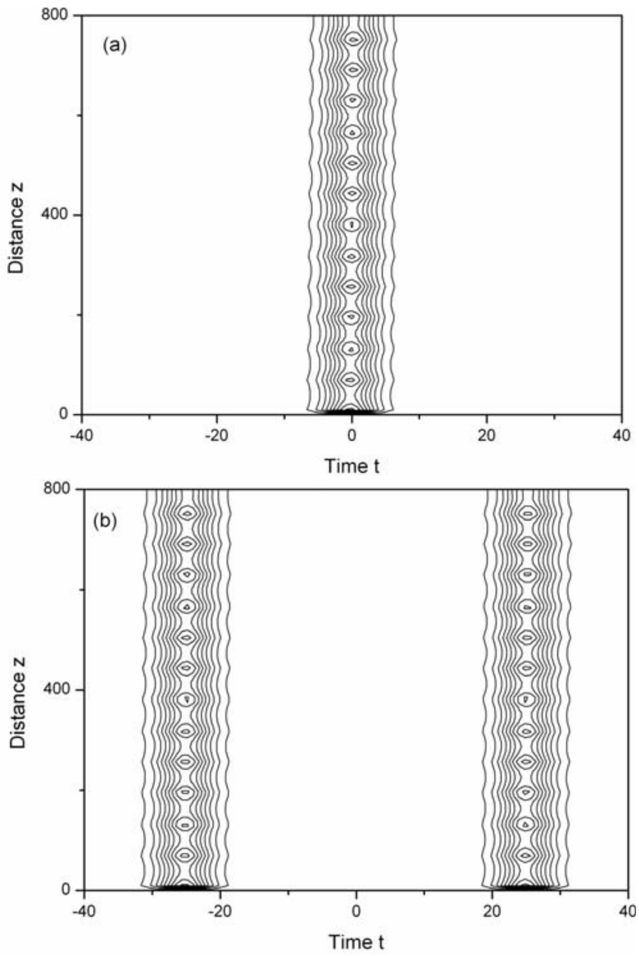

Fig. 3. Contour plots of the one- and two-soliton solution when $g = 0$, the other parameters are the same as those in Figures 1 and 2.

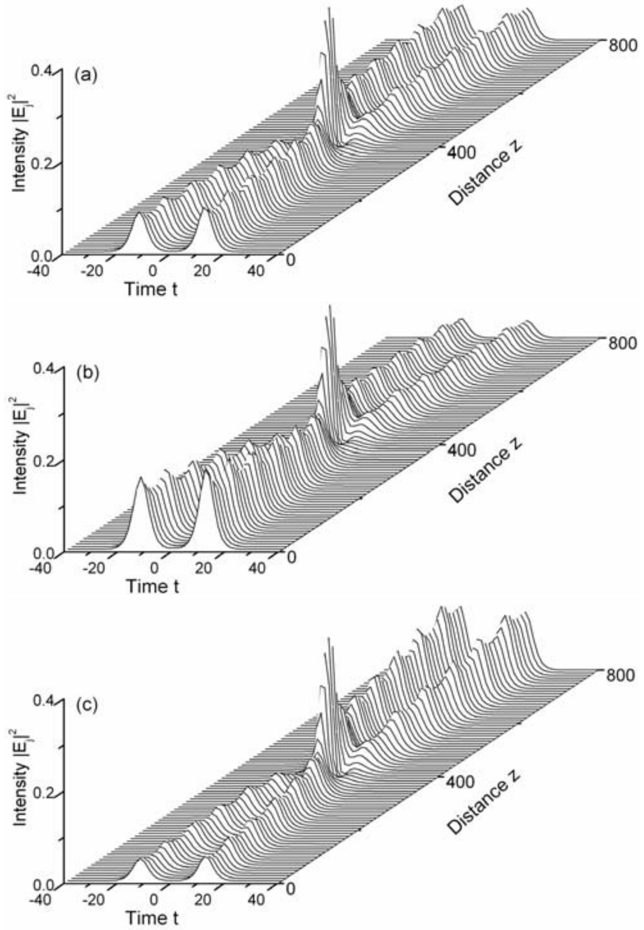

Fig. 4. Pulse evolution plots of elastic collision of two-soliton solution when $d = -1$, $f = -2$, $\sigma = 2$, $T_0 = 0$, $\xi_1 = 0.560$, $\eta_1 = 0.2$, $\xi_2 = 0.562$, $\eta_2 = 0.2$, $\chi_1 = \chi_2 = \sqrt{2}/2$, $c_1 = 0.4$, $c_2 = 0.5$, and (a) $g = 0$, (b) $g = -0.03$, (c) $g = 0.03$.

solution when both the solitons are infinitely apart. This may be achieved by taking the limit $\vartheta_k \to \pm\infty$. When we set $\vartheta_2 \to +\infty$, the first component of the two-soliton solution has the form

$$E_j^{1+} \to \sqrt{b_1 b_2}\chi_j \sqrt{\frac{D_2}{F}} \frac{(a_1 + ia_2)\exp(2i\varphi_1)}{\cosh(2\vartheta_1 + R)}, \qquad (18)$$

where $j = 1, 2$ and $\exp(R) = \sqrt{b_1/b_2}$. When we set $\vartheta_2 \to -\infty$, the first component of the two-soliton solution has the form

$$E_j^{1-} \to \sqrt{b_1 b_2}\chi_j \sqrt{\frac{D_2}{F}} \frac{(a_1 - ia_2)\exp(2i\varphi_1)}{\cosh(2\vartheta_1 - R)}. \qquad (19)$$

Thus we can define the amplitude's transition matrix for the first component of the two-soliton solution as

$$T_j^1 = \frac{a_1 + ia_2}{a_1 - ia_2}. \qquad (20)$$

Similarly, when we set $\vartheta_1 \to +\infty$, the second component of the two-soliton solution has the form

$$E_j^{2+} \to \sqrt{b_1 b_2}\chi_j \sqrt{\frac{D_2}{F}} \frac{(a_3 - ia_2)\exp(2i\varphi_2)}{\cosh(2\vartheta_2 + R)} \qquad (21)$$

and when we set $\vartheta_1 \to -\infty$, the second component of the two-soliton solution has the form

$$E_j^{2-} \to \sqrt{b_1 b_2}\chi_j \sqrt{\frac{D_2}{F}} \frac{(a_3 + ia_2)\exp(2i\varphi_2)}{\cosh(2\vartheta_2 - R)}. \qquad (22)$$

Also we can define the amplitude's transition matrix for the second component of the two-soliton solution as

$$T_j^2 = \frac{a_3 - ia_2}{a_3 + ia_2}. \qquad (23)$$

We can easily understand from equations (17) and (20) that if $|T_j^l| \neq 1$, $(l = 1, 2)$, there will have energy exchanges at the time of interaction [25]. Other wise, the solitons will pass through each other without being affected in their shapes and sizes when the collision happens. The phase shift as a result of collision may be obtained as $2R = \ln(b_1/b_2)$. Here in this paper, one can easily find that $|T_j^l| = 1$, $(l = 1, 2)$ is always satisfied, namely, the later case was always satisfied. The evolution plots were illustrated in Figure 4. We can see from the figure that the two components of the two-soliton solution surely propagate

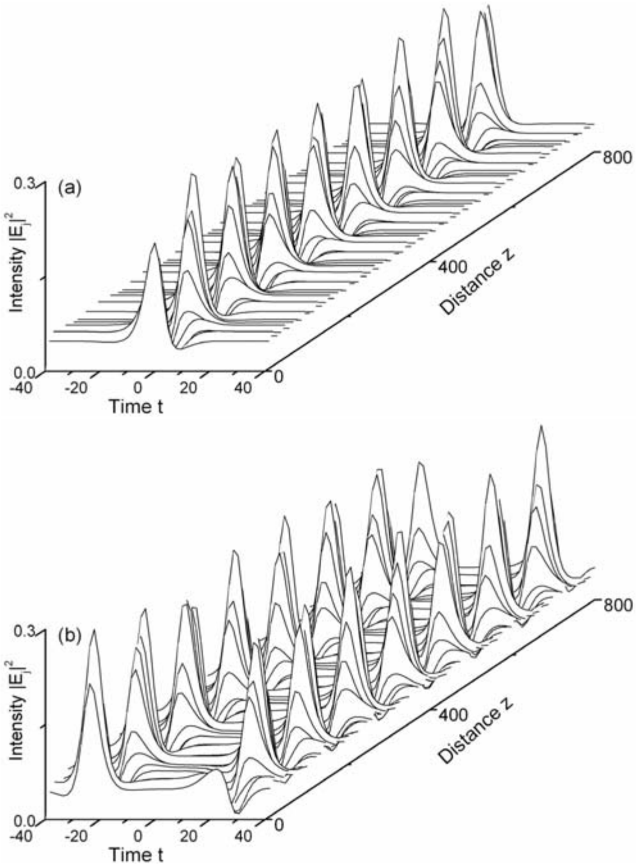

Fig. 5. Pulse evolution plots of (a) one-soliton solution, (b) two-soliton solution under the perturbation of CW solution whose amplitude is $A_{jcw} = 0.3$. The other parameters are the same as those in Figure 2.

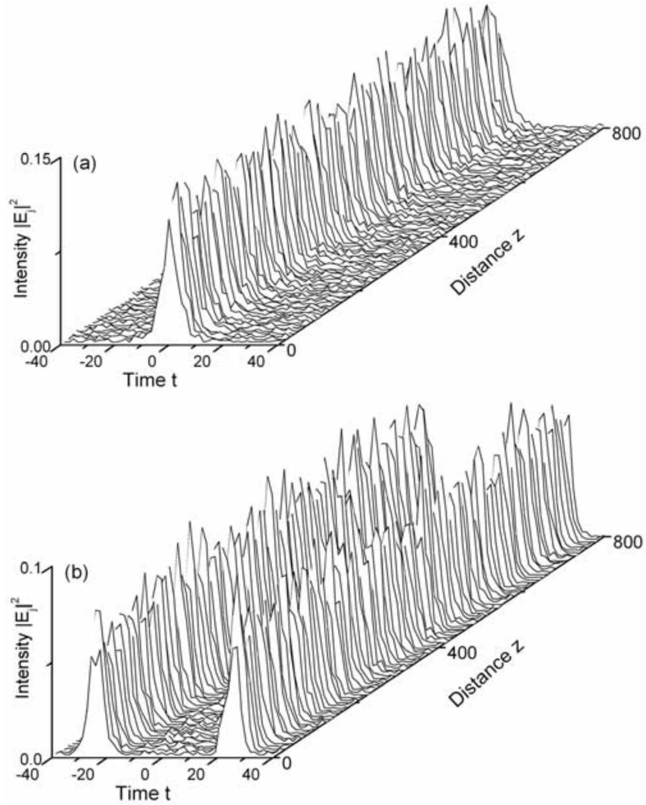

Fig. 6. Pulse evolution plots of (a) one-soliton solution, (b) two-soliton solution under the perturbation of white noise whose maximum amplitude is 0.1. The other parameters are the same as those in Figure 2.

steadily without any energy exchanges even though collision happened.

It is worth noting that the existence of the soliton solutions obtained above depends on the specific nonlinear and dispersive features of the medium, which have to satisfy the condition (4) from which equation (1) can be transformed to the well-known integrable coupled Hirota equation. This constraint conditions present the strict balances among the model parameters. In real applications, however, it may be difficult to produce exactly such balances. Therefore the study of some cases under the perturbations is also necessary. Here we provide two types of further researches, one is continuous wave (CW) perturbation and the other is white noise perturbation. For the given equation (1), we have found that there is a CW solution, which is of the form

$$E_{jcw} = A_{jcw}\sqrt{\frac{D_2(z)}{F(z)}}\exp\left(i\frac{1}{6}t + i\frac{1}{108}\int D_2(z)\,dz\right). \tag{24}$$

So, we take the initial pulse as $E_j = E_{js} + E_{jcw}$ to do the numerical simulations, where E_{js} is the exact soliton solution and E_{jcw} is the CW solution. Generally, E_j is not an exact solution of equation (1), but since we are interested in the evolution of the superposition of a bright soliton pulse and CW solution, the solution of equation (1) been written as the sum of E_{js} and E_{jcw} is possible [26]. The evolution plots were illustrated in Figure 5a for one-soliton solution and Figure 5b for two-soliton solution. The evolution plots when white noise whose maximum amplitude is 01. is added were illustrated in Figures 6a and 6b, respectively. From both Figures 5 and 6 we can see that the pulses are still stable when propagating in optical fibers.

3 Conclusions

In conclusion, by using of Darboux transformation, we have solved a family of coupled higher-order nonlinear Schrödinger equation with variable coefficients, which is always used to describe the optical soliton propagating in inhomogeneous optical fiber media and exact N-soliton solution is obtained in detail. The solutions' characteristics of stabilities and collisions are discussed analytically and numerically under a given soliton control system. Our results are of special application in short optical soliton propagation systems and the further research should be an interesting task.

Appendix

Here we give the standard Darboux transformation to solve the equation (1).

Rewriting

$$U = \begin{pmatrix} -i\lambda & mE_1 & mE_2 \\ -mE_1^* & i\lambda & 0 \\ -mE_2^* & 0 & i\lambda \end{pmatrix} = i\lambda J + P \quad \text{(A.1)}$$

where

$$J = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad P = \begin{pmatrix} 0 & mE_1 & mE_2 \\ -mE_1^* & 0 & 0 \\ -mE_2^* & 0 & 0 \end{pmatrix}.$$

The linear eigenvalue problem is given as

$$\Phi_t = U\Phi,\, \Phi_z = V\Phi,\, \Phi = (\phi_1,\, \phi_2,\, \phi_3)^T. \quad \text{(A.2)}$$

Introducing the transformation

$$\Phi' = (\lambda I - S)\Phi, \quad S = H\Lambda H^{-1}, \quad \Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3) \quad \text{(A.3)}$$

where H is a nonsingular matrix, requiring

$$\Phi'_t = U'\Phi', \quad U' = i\lambda J + P' \quad \text{(A.4)}$$

where

$$P' = \begin{pmatrix} 0 & mE'_1 & mE'_2 \\ -mE_1^{'*} & 0 & 0 \\ -mE_2^{'*} & 0 & 0 \end{pmatrix}.$$

Then combining equations (A.1–A.4), we can obtain the Darboux transformation for equation (1) in the form

$$P' = P + JS - SJ. \quad \text{(A.5)}$$

Let

$$\Phi_t^{(j)} = U_1\Phi^{(j)}, \quad (j = 2, 3), \quad U_1 = i\lambda^* J + P$$

where $\Phi^{(j)} = (\phi_1^{(j)},\, \phi_2^{(j)},\, \phi_3^{(j)})^T$. Now we can set $H = (\phi_k,\, \phi_k^{(2)},\, \phi_k^{(3)})$. Write

$$\Delta_1^1 = \begin{vmatrix} \phi_2^{(2)} & \phi_2^{(3)} \\ \phi_3^{(2)} & \phi_3^{(3)} \end{vmatrix},\ \Delta_1^2 = \begin{vmatrix} \phi_3^{(2)} & \phi_3^{(3)} \\ \phi_1^{(2)} & \phi_1^{(3)} \end{vmatrix},\ \Delta_1^3 = \begin{vmatrix} \phi_1^{(2)} & \phi_1^{(3)} \\ \phi_2^{(2)} & \phi_2^{(3)} \end{vmatrix}.$$

By straightforward calculation, we can find that $\exp(i\lambda t)\left(\Delta_1^{1*},\, \Delta_1^{2*},\, \Delta_1^{3*}\right)^T$ satisfies $\Phi_t = U\Phi$. So we can choose Φ in H as above and have $\Delta_1^j = \exp(-i\lambda^* t)\phi_j^*$. Then

$$Q = \det H = \sum_{j=1}^{3} \phi_j \Delta_1^j = \exp(-i\lambda^* t)\sum_{j=1}^{3}\phi_j\phi_j^* = \exp(-i\lambda^* t)\Delta.$$

Setting $\lambda_1 = i\lambda$, $\lambda_2 = \lambda_2 = i\lambda^*$, and by the definition of S, we have

$$S = \begin{pmatrix} i\lambda^* + i\frac{\lambda-\lambda^*}{\Delta}\phi_1\phi_1^* & i\frac{\lambda-\lambda^*}{\Delta}\phi_1\phi_2^* & i\frac{\lambda-\lambda^*}{\Delta}\phi_1\phi_3^* \\ i\frac{\lambda-\lambda^*}{\Delta}\phi_2\phi_1^* & i\lambda^* + i\frac{\lambda-\lambda^*}{\Delta}\phi_2\phi_2^* & i\frac{\lambda-\lambda^*}{\Delta}\phi_2\phi_3^* \\ i\frac{\lambda-\lambda^*}{\Delta}\phi_1^*\phi_3 & i\frac{\lambda-\lambda^*}{\Delta}\phi_2^*\phi_3 & i\lambda^* + i\frac{\lambda-\lambda^*}{\Delta}\phi_3^*\phi_3 \end{pmatrix}.$$

Thus from equation (A.5), we can obtain the other solutions for equation (1) as

$$R'_1 = R_1 - 2\sqrt{\frac{D_2}{F}}S_{12}, \quad R'_2 = R_2 - 2\sqrt{\frac{D_2}{F}}S_{13}. \quad \text{(A.6)}$$

Taking the Darboux transformation N times, we can find the N-soliton solutions for equation (1) as expressed in equation (7).

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